T-norm-based fuzzy logics and logics for reasoning under vagueness

Thomas Vetterlein, Klaus-Peter Adlassnig

Institute for Medical Expert and Knowledge-Based Systems, Medical University of Vienna
Vienna, Austria
Email: {Thomas.Vetterlein,Klaus-Peter.Adlassnig}@meduniwien.ac.at

Abstract— *We contrast the concept underlying t-norm-based propositional fuzzy logics with the problem to whose solution fuzzy logics are frequently suggested as helpful – namely, to find a model of reasoning with vague information. We argue that fuzzy logics are useful as long as truth values can be identified with the meaning of the considered propositions. This, however, is rarely the case in practice; hence we see the need to broaden the concept underlying this important class of logics and try fresh approaches. In particular, we should flexibilise the formalism to allow that propositions do not arise in the same context, but are just known to be related in some way.*

We tackle the problem tentatively. We define a set of rules which, as we assume, are minimally required to enable us to argue about vague propositions whose content is not taken into account. Our choice of rules reflects the practical requirements of a certain expert system on which we work.

Although we deal here with fuzzy logic in a very direct sense, we arrive at calculi completely different from the t-norm-based ones. Without incorporating truth degrees explicitly, we are led to Belnap's logic, which can, but need not, be endowed with a semantics based on graded truth degrees. When formalising also truth degrees, we get a logic which can be based on what we call metric De-Morgan lattices.

Keywords— t-norm-based fuzzy logics, reasoning under vagueness, medical expert system, De-Morgan lattices, metric De-Morgan lattices

1 T-norm-based fuzzy logic for reasoning about vague information – a trap

Fuzzy logics are distinguished from classical logic by the incorporation of an extended set of truth values. In the standard case, the value 0 is used to express falsity, the value 1 is used to express trueness, and all remaining real values in between these two limit points are added in order to cope with the fact that objects may fulfil a property to an intermediate degree. We arrive naturally at the idea to evaluate propositions in the real unit interval, whose most basic feature is its linear order.

The connectives used in fuzzy logics consequently need to be interpreted by operations on $[0, 1]$. Typically, a conjunction is present, which is typically interpreted by a left-continuous t-norm. Sometimes, an involutive negation is present as well, conveniently interpreted by the standard negation $1 - \cdot$. A further connective, which logicians, in contrast to engineers, generally consider as the most basic one, is the implication, which is typically interpreted by the residuum belonging to the t-norm. Finally, the logic may or may not offer the possibility to express explicitly to which degree a proposition holds.

Based on this approach, more than just a few logics have been defined and intensively studied. Monographs of basic importance include [14, 13] as well as, as regards the explicitation of truth degrees, [18]. We remark that we certainly do not address all logics which have been called "fuzzy" in the literature; this would be impossible as it seems that nowadays any logic has a "fuzzy" counterpart. Here, we just speak about logics of the indicated type.

We wish to address in this note the peculiar relationship between t-norm-based fuzzy logics and a problem which is regularly mentioned in discussions on the fundamentals of fuzzy logics: how to formalise reasoning when the referred information is possibly vague. The discussion on the very nature of fuzzy logics is old. An aim has been to develop fuzzy logics, as they are, from clear, meaningful principles; see, among many papers, e.g. [9, 10, 21, 22]. Here, we want to approach once the subject from the other side, namely, from the point of view of a specific application: an appropriate formalisation of justifiable reasoning can simply be a practical need.

Let us first try to formulate our concern in an abstract way. We assume to be given a set of propositions whose content does not matter. This means that for the derivation of consequences, the meaning is not taken into account and can be assumed to be unknown. We just know that the propositions refer to the observable or unobservable properties of somebody or something, describable in, possibly scientific, natural language. Important for us, the propositions express the presence of some property which can be vague, where, as usual, vagueness is characterised by the possibility of borderline cases. Moreover, we assume to have some knowledge about the mutual relationships. These relationships may express that some property is more general than another one, or a causal implication based on experience; again, we do not require the relationships to hold necessarily strict. What we are finally interested in, is to find a formal framework which tells us how to derive new information from the one we have to our disposal. Since the outlined situation is very general, the framework must be very general as well; we wonder, so to say, about a minimal logic for reasoning under vagueness.

The described problem is not purely academic and in particular not part of any "ivory-tower" philosophical theory, but has a practical background. We work towards an appropriate formalisation of a medical expert system [7]. The system which we are to analyse is called Cadiag-2, the second generation of the expert systems Cadiag – "*C*omputer-*A*ssisted *Diag*nosis", and aims at the differential diagnostic decision support in patient care [1, 2]. Cadiag-2 processes both vague and uncertain data; we restrict here to the first case only. The procession of uncertain data calls for a probabilistic logic and causes completely different problems than those discussed here; for probability theory, well-founded calculi exist, whose unpleas-

ISBN: 978-989-95079-6-8

ant feature, however, can be a too high complexity as well as incomprehensible inference rules.

As we intend to formalise propositions disregarding their content, our problem is clearly a case for a propositional logic. As mentioned, many propositional fuzzy logics have been proposed in the past, based on different ways how to endow the real unit interval with a structure. Moreover, a wide range of logics has been introduced related to the problem which we address here, in particular different versions of "logics of argumentation". For a comprehensive overview and a large collection of references, we recommend the handbook [12]. Here, we just mimic the first steps towards the generally much more sophisticated and often more specialised systems found in the literature. What we have in mind is to concentrate exclusively on the aspect of vagueness, to proceed in a way which can hardly be further generalised, and to see how the result relates to t-norm-based fuzzy logics.

The set of eligible propositional logics is much restricted by one basic requirement, dictated by the intended application: all constituents of the formal logic need to have a counterpart on the informal level. Specifically, there must exist a plausible way how to think about each connective which appears in the logic's language, most easily obtained by a clear correlation with a natural-language expression. Moreover, if a proposition is provable from others, there must exist a proof in a proof system such that each step is comprehensible as a plausible argument, rather than a pure manipulation of strings; in the best case, each step can easily be translated to an explanation in natural language, exhibiting the causal or logical relationship on which the argument is based.

A fuzzy logic of the above-indicated kind does not meet any of these requirements. To see the problem, let us assume that we actually can find a logic fitting to our needs. We actually feel that it is natural to assume so; after all, we wish to formalise possibly vague statements, and these are appropriately evaluated on a linear continuous scale. So let us see how the logic could look like. We need a conjunction $\wedge$ and a negation $\sim$; the interpretation by the infimum and the standard negation, respectively, will do in our case. We note that the interpretation of truth values is not our subject here, and our particular choices for the connectives provide just an example. Moreover, we need to express truth degrees explicitly; to this end, we add constants $\bar{r}$ for each rational $r \in [0,1]$. So far, we do not encounter problems.

The first serious problem comes with the implication. Let us tentatively add the connective $\rightarrow$ interpreted by the residuum belonging to $\wedge$. Note that we then arrive at the logic $\mathsf{RGL}_\sim$, the Gödel logic enriched with the standard negation and truth constants [8]. Now, there is a natural way how to think about a statement "$\alpha \rightarrow \beta$": we interpret it as "α implies β". However, this clarity disappears as soon as we nest implications on the left side, like in "$(\alpha \rightarrow \beta) \rightarrow \gamma$". If our reference, a set of propositions, has a priori the structure of a residuated lattice, we can say that $\alpha \rightarrow \beta$ denotes the weakest element which, together with α, implies β. But in our case, there is no such structure available; we recall that we do not wish to assume any a priori structure as we would have to analyse the propositions by content then.

We note that this critics is related to the discussion out of which relevance logics arose [3]. Furthermore, a discussion in which the role of the implication connective in fuzzy logics is opposed to the needs of certain applications, can be found in [4].

For us, there is a reasonable way how to proceed: to drop the implication as a generally applicable connective. To this end, we consider in [7] a logic in which the implication always appears at the only place at which it can be appropriately called an implication: on the outermost level. Namely, we consider pairs of implication-free formulas, with the intended meaning that the left one denotes a proposition which is stronger then the right one. Thus we leave the area of t-norm-based logics and enter the field of lattice word problems. Namely, our model is the following algebra K, a Kleene algebra with added constants: $([0,1]; \wedge, \vee, \sim, (\bar{r})_{r \in \mathbb{Q}[0,1]})$.

We arrive at a logic, which we call GZL, whose formal expressions possess straightforward interpretations. However, we encounter the second and even more serious problem when considering a proof system for GZL. Following the lines of [5], we have in [7] presented a proof system based on sequents-of-relations. Unfortunately, our requirement that proof steps should be comprehensible in an informal way, is far from being fulfilled. Consider the K-tautology

$$\alpha \wedge \sim\alpha \;\rightarrow\; \beta \vee \sim\beta. \tag{1}$$

In its proof, we have to make necessarily use of the possibility to use multisets of relations, namely,

$$\alpha \Rightarrow \beta \;\mid\; \beta \Rightarrow \alpha \tag{2}$$

will appear in the proof. However, α and β refer to arbitrary facts, and the tentative translation "α implies β, or β implies α" is nonsense. Note that the problem concerns the proved result as well; (1) cannot be interpreted as a statement which anybody would ever tell.

The deeper reason for this difficulty is the semantics. An element of $[0,1]$ is intended to be the truth degree of a proposition; but it is treated like its meaning. What we might think as being associated to a property, telling that the property does not fully apply, is already the property itself. As a consequence, when using $[0,1]$ as a model, we may be led to the situation that we compare something by strength what by content would never be comparable. A valid statement of the form "$\alpha \rightarrow \beta$" is not really translatable to "from α we can conclude β", but only that α is under all circumstances assigned a smaller truth value than β, and based on this interpretation, (2) becomes indeed meaningful and just expresses the linear order of the truth degrees. However, this interpretation is not what we want.

The interpretation in the linearly ordered set of reals may certainly be useful at other places. A requirement comes into play which we have frequently argued for: to put fuzzy logics on firm grounds, we need first to be aware of the nature of what we reason about. In case of t-norm-based fuzzy logic, we reason about a set of propositions which has the internal structure of a residuated lattice, as it is the case for universes of fuzzy sets. The same, by the way, applies to classical propositional logic, which exactly reasons about a collection of propositions endowed a priori with the structure of a Boolean algebra; the popular claims about a "general validity" of this logic are meaningless.

ISBN: 978-989-95079-6-8

For us, the only way out is to restrict the calculus for GZL to those inferences which are not in conflict with our intended interpretation: the sequent $\alpha \Rightarrow \beta$ should mean that α is a statement stronger than β. We can achieve this is by not allowing multisets of relations, but only single relations. The interesting observation in [7] is that by means of this restriction, we get – not exactly but, say – very close to the logic which is actually used for the expert system which we examine.

The observation that the logic which we need arises by a certain restriction of a t-norm-based fuzzy logic, might be considered interesting, but not really satisfying. It rather suggests that the conceptual differences between the logic underlying systems like Cadiag-2 or similar expert systems on the one hand, and fuzzy logics on the other hand, cannot be bridged.

2 A minimal logic for reasoning under vagueness, without explicit degrees

The problem how to formalise ways to argue about vague propositions of unspecified content and their mutual interrelations, calls for alternative solutions. Let us opt for the syntactical approach; we will assemble some inference rules which translate to argumentation steps in a straightforward way. We will then check if some semantics with a reasonable interpretation can be found ex post, taking all imaginable possibilities into account and in particular not restricting ourselves to structures known from fuzzy logics or fuzzy set theory.

We note that this procedure seems to be in sharp contrast to the guiding principles of mathematical modelling which we have defended earlier, namely the principle that prior to any formalisation, the structure of reference needs to be specified first, in a way that the meaning of all its constituents is clear. However, in the present case, we do not do metamathematics, we do not examine ways how structures of a certain type are generally examined in a sound way; we do mathematics. Namely, it is the way of reasoning itself which is our object of investigation, and we do not share the opinion that rules for proper argumentation are fixed and thus can be derived from some higher-level truth. Intuitively acceptable inference rules will rather constitute a structure over a set of atomic propositions, and we do not assume a canonical answer how it may look like. In any case, we examine a logic as a mathematical object, the notion "logic" just being a name for it.

As indicated, there is not really a canonical way to select rules. One may argue against certain rules shown below, or feel that there is something missing. A discussion would not be fruitful if no guidelines were provided. We keep with the application in medicine; the rules shown below are extracted from those essential for the medical expert system with whose formalisation we are concerned.

In this section, we consider the case that we do not deal with truth values explicitly. We define the propositional logic DML as follows.

Definition 2.1 The *propositions* of DML are built up from a set of symbols $\varphi_1, \varphi_2, \ldots$ and the two constants $\bar{0}, \bar{1}$ by means of the binary connectives $\wedge, \vee$ and the unary connective $\sim$; the set of propositions is denoted by $\mathcal{F}_L$. The *implications* of DML are ordered pairs of propositions, denoted by $\alpha \to \beta$, where $\alpha, \beta \in \mathcal{F}_L$; the set of implications is denoted by $\mathcal{F}_I$.

Moreover, a *sequent* is an ordered pair of a non-empty finite set of propositions and a single proposition, notated by $\gamma_1, \ldots, \gamma_k \Rightarrow \delta$. The axioms and rules of DML are the following, for any propositions α, β, γ and sequent Γ:

$$\bar{0} \Rightarrow \alpha \qquad \alpha \Rightarrow \alpha \qquad \alpha \Rightarrow \bar{1}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha \Rightarrow \beta}{\Gamma \Rightarrow \beta} \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma, \beta \Rightarrow \alpha}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \qquad \frac{\Gamma, \alpha, \beta \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta \Rightarrow \gamma}$$

$$\frac{\Gamma, \alpha \Rightarrow \gamma \quad \Gamma, \beta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta \Rightarrow \gamma} \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}$$

$$\frac{\alpha \Rightarrow \beta}{\sim\beta \Rightarrow \sim\alpha} \qquad \frac{\sim\alpha \Rightarrow \beta}{\sim\beta \Rightarrow \alpha} \qquad \frac{\alpha \Rightarrow \sim\beta}{\beta \Rightarrow \sim\alpha}$$

The notion of a proof of a sequent from a finite set of sequents is defined in the expected way. A *theory* of DML is a finite set of implications. An implication $\alpha \to \beta$ is called provable from $\mathcal{T} = \{\alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n\}$ if there is a proof of $\alpha \Rightarrow \beta$ from $\{\alpha_1 \Rightarrow \beta_1, \ldots, \alpha_n \Rightarrow \beta_n\}$, in signs $\mathcal{T} \vdash \alpha \to \beta$.

As to be expected, a sequent $\gamma_1, \ldots, \gamma_k \Rightarrow \delta$ is meant to express that "γ_1 and ... and γ_k imply δ".

Note that this logic again does not contain the implication as a connective, and it does contain a negation. Moreover, the calculus is sound, but not complete, with respect to classical two-valued interpretations; again (1) is not derivable. For the lack of the implication connective, also the comparison with intuitionistic logic is (probably) not well possible.

But a semantics is easily found, since DML is Belnap's logic of De-Morgan lattices; DML differs only slightly from the calculus presented in [11]. A structure $(M; \wedge, \vee, \sim, 0, 1)$ is a De-Morgan lattice if (i) $(M; \wedge, \vee, 0, 1)$ is a distributive lattice and (ii) $\sim$ is an order-reversing and involutive unary operation. De-Morgan lattices are subalgebras of direct products of the algebra $(\mathcal{M}_4; \wedge, \vee, \sim, 0, 1)$, where $(\mathcal{M}_4; \wedge, \vee, 0, 1)$ is the four-element Boolean lattice and $\sim$ maps each of the two atoms to itself [16]. It follows that we can provide a semantics based on $\mathcal{M}_4$; we note that assigning one of the four truth values to a proposition φ is usually interpreted as that φ is known to be true, false, neither true nor false, both true and false, respectively.

Most remarkably, fuzziness does not appear. However, the algebra $\mathcal{M}_4$ possesses a natural "fuzzified" extension, and we may alternatively base DML on a kind of fuzzy semantics. To make the comparison possible, we consider the pair V_c and V_f; note that the algebra V_c is isomorphic to $\mathcal{M}_4$.

Definition 2.2 Let $V_c = \{(s,t) : s,t \in \{0,1\}\}$, endowed with the componentwise natural order and the operation $\sim$ defined by

$$\sim (s,t) \;=\; (1-t, 1-s) \tag{3}$$

for $s, t \in \{0,1\}$. A *crisp evaluation* of DML is a mapping $v \colon \mathcal{F}_L \to V_c$ preserving $\wedge, \vee, \sim$ and mapping $\bar{0}$ to $(0,0)$ and $\bar{1}$ to $(1,1)$. An implication $\alpha \to \beta$ is then said to be *satisfied* by v if $v(\alpha) \leq v(\beta)$. A theory $\mathcal{T}$ is said to *crisply entail* an implication $\alpha \to \beta$ if the latter is satisfied by all crisp evaluations satisfying every element of $\mathcal{T}$; we write $\mathcal{T} \models_c \alpha \to \beta$ in this case.

ISBN: 978-989-95079-6-8

Furthermore, let $V_f = \{(s,t)\colon\ s,t \in [0,1]\}$, endowed with the componentwise natural order and the operation $\sim$, which is again defined by (3), where however this time $s,t \in [0,1]$. We define *fuzzy evaluations*, *satisfaction*, and *fuzzy entailment* similarly as above.

Theorem 2.3 *Let $\mathcal{T}$ be a theory and $\alpha \to \beta$ an implication of* DML. *Then $\mathcal{T} \vdash \alpha \to \beta$ if and only if $\mathcal{T} \models_c \alpha \to \beta$ if and only if $\mathcal{T} \models_f \alpha \to \beta$.*

Proof. Completeness in the indicated sense, but with respect to arbitrary De-Morgan lattices, holds due to [11, Theorem 4.11, (A1)]. But any De-Morgan lattice is a subalgebra of a direct product of copies of V_c, or alternatively, of a direct product of copies of V_f. □

As regards the interpretation of the semantics which we have proposed for DML, the situation is surprising with respect to the fuzzy variant. There is a close connection of DML to a logic which has been proposed in the context of decision making, based on the observation that we often consider separately the arguments in favour and the arguments against a possible decision [19].

On the other hand, the fact that we can work, without the need to change the inference rules, with crisp truth values as well, is somewhat disillusioning. The reasoning is the same if we assume our propositions to refer to vague or crisp properties. The situation certainly changes when we include truth degrees, as to be done next.

3 A minimal logic for reasoning under vagueness, with explicit degrees

The logic DML discussed in the last section is intended to be useful for reasoning about relationships between statements involving vagueness. The vagueness, however, cannot be addressed directly; and the logic can actually equally well considered as a logic not concerning vagueness. Moreover, a possible non-strictness of the relationships themselves is not expressible.

In applications, it can be desirable to have the possibility to denote a vague property by one single symbol, to which the degree of presence is explicitly attached. In this chapter, we attempt to formulate a calculus similarly to DML, but with explicit reference to truth degrees.

This problem is much more involved. We believe that there are many possibilities, and that the decision which is the best is even more difficult than in the above case.

According to a common procedure, we could enrich the language by truth constants. We will not follow this way; apart from the fact that we have not succeeded to produce a reasonable result concerning a logic DML enriched with truth constants, the idea is actually not well in accordance with the guidelines formulated above: truth constants should not be mixed with the meaning of a proposition. So we will use the two sharp truth constants only, representing falsity and trueness.

Truth constants should appear on a separate level. We propose to make a graded implication the basic syntactical constituent:

$$\alpha \xrightarrow{t} \beta,$$

with the intended meaning that α implies β to a degree $\geq t \in [0,1]$, where α and β are propositions of DML. As an example, let α denote a crisp proposition like "having a body temperature of $37.8°C$", and let β denote "having fever"; then the statement would hold with, say, $t = 0.8$.

In general, α and β are meant to refer to any vague property. Then either t refers to the compatibility of α with β. Alternatively, we can mean that α and β are causally related; then the smaller t is, the less strict is this relationship. Finally, we may also deal with single properties. Namely, the expressions

$$\bar{1} \xrightarrow{t} \delta \quad \text{and} \quad \delta \xrightarrow{t} \bar{0} \tag{4}$$

may serve to express that δ holds to the degree t, or δ is refused with the degree t, respectively.

To formulate inference rules is not straightforward; in the present context, our aim can only be plausibility. The basic question is which truth degree is, by tendency, assumed after two successive inference steps which are both based on a non-strict relationship. To examine this problem is even more difficult than to make a reasonable choice with regard to the truth values themselves, a problem which has been studied numerous times, see e.g. [15, 17]. It would clearly be desirable to have methods at hand to examine also the present situation empirically, in analogy to the procedure followed in [15].

Only the case of implications of the form $\alpha \xrightarrow{1} \beta$ is clear; they are supposed to express strict relationships, and for them, the rules of DML should be applicable. Furthermore, the degrees are assumed to be lower bounds, hence an implication is the less expressive the smaller the indicated truth degree is. Statements of the form $\alpha \xrightarrow{0} \beta$ do not express anything.

So let us see how the set of rules for our refined logic could look like. For implications $\alpha \xrightarrow{1} \beta$, we will use the rules of DML. For implications of the form $\alpha \xrightarrow{t} \beta$, where $t < 1$, the rules of DML introducing $\wedge$ or $\vee$ are not generalisable though; the degree of the conclusion cannot be assumed to be calculable from the degrees of the assumptions. However, what we should be able to say is, if we replace α by a stronger proposition, or β by a weaker proposition, then the relationship between α and β should be characterised by a higher truth value, so that t will still be a lower bound. Next, assume that we have proved $\alpha \xrightarrow{s} \beta$ and $\beta \xrightarrow{t} \gamma$; then $\alpha \xrightarrow{u} \gamma$ will be derivable as well, and we have to offer a way to calculate the degree u from s and t. As mentioned above, a well-founded decision is impossible, hence just like in case of the design of a fuzzy logic, a pragmatic solution is needed here. We opt for the operation dual to the truncated addition: we take here the Łukasiewicz t-norm $\odot\colon\ [0,1]^2 \to [0,1],\quad (a,b) \mapsto (a+b-1) \vee 0$.

We specify the propositional logic ArgL as follows.

Definition 3.1 The set $\mathcal{F}_L$ of *propositions* is defined like for DML. An *implication* of ArgL consists of ordered triples of two propositions and a rational value $t \in [0,1]$; we write $\alpha \xrightarrow{t} \beta$, where $\alpha, \beta \in \mathcal{F}_L$; the set of implications is denoted by $\mathcal{F}_I$.

Moreover, a *sequent* is an ordered triple consisting of a nonempty finite set of propositions, a single proposition, and a rational value $t \in [0,1]$; we write

$$\gamma_1, \ldots, \gamma_k \overset{t}{\Rightarrow} \delta.$$

ISBN: 978-989-95079-6-8

The *crisp* rules of ArgL are those of DML, the symbol $\Rightarrow$ being replaced at all places by $\stackrel{1}{\Rightarrow}$.

The *fuzzy* rules of ArgL are the following:

$$\frac{\Gamma \stackrel{s}{\Rightarrow} \alpha \quad \alpha \stackrel{t}{\Rightarrow} \beta}{\Gamma \stackrel{s \odot t}{\Rightarrow} \beta} \quad \frac{\Gamma \stackrel{t}{\Rightarrow} \alpha}{\Gamma \stackrel{s}{\Rightarrow} \alpha}, \text{ where } s \leq t$$

$$\frac{\Gamma, \alpha \stackrel{t}{\Rightarrow} \delta}{\Gamma, \alpha \wedge \beta \stackrel{t}{\Rightarrow} \delta} \quad \frac{\Gamma \stackrel{t}{\Rightarrow} \alpha}{\Gamma \stackrel{t}{\Rightarrow} \alpha \vee \beta}$$

The notion of a *proof*, a *theory*, the *provability* of an implication from a theory, is defined similarly like for DML.

To associate to this calculus a reasonable semantics, is the next challenge. Only one point seems to be certain – we are not led to fuzzy sets. The only remarkable fact is that a t-norm is involved; for connections between t-norms and a somewhat similar setting, see [6, 20].

Let us consider the following structures, so-to-say the algebraic counterpart of ArgL. Here, $\oplus : [0,1]^2 \to [0,1],\ (a,b) \mapsto (a+b) \wedge 1$ is the t-conorm associated to $\odot$.

Definition 3.2 A structure $(A; \wedge, \vee, \sim, 0, 1, d)$ is called a *metric De-Morgan lattice* if $(A; \wedge, \vee, \sim, 0, 1)$ is a De-Morgan lattice and $d\colon A \times A \to [0,1]$ is such that (i) $d(a,b) = 0$ if and only if $a \leq b$ and (ii) $d(a,c) \leq d(a,b) \oplus d(b,c)$.

An *evaluation* of ArgL in a metric De-Morgan lattice A is a mapping $v : \mathcal{F}_L \to A$ preserving $\wedge, \vee, \sim$ and the constants. An implication $\alpha \stackrel{t}{\to} \beta$ is *satisfied* by an evaluation v if $d(v(\alpha), v(\beta)) \leq 1 - t$. Semantic entailment is defined as usual.

Let us consider the following instructive example. Let $(A; \wedge, \vee, \sim, 0, 1)$ be a Boolean algebra, and let $\mu\colon A \to [0,1]$ be a strictly positive submeasure on A, meaning that, for $a, b \in A$, (i) $\mu(0) = 0$, (ii) $\mu(a) > 0$ if $a > 0$, (iii) $a \leq b$ implies $\mu(a) \leq \mu(b)$, (iv) $\mu(1) = 1$, and (v) $\mu(a \vee b) \leq \mu(a) \oplus \mu(b)$. Furthermore, put $d(a,b) = \mu(a \wedge \sim b)$. Then we may check that $(A; \wedge, \vee, \sim, 0, 1, d)$ is a metric De-Morgan lattice.

Theorem 3.3 *Let $\mathcal{T}$ be a theory and $\alpha \stackrel{t}{\to} \beta$ an implication of* ArgL*, where $t \in (0,1]$. Then $\mathcal{T} \vdash \alpha \stackrel{t}{\to} \beta$ if and only if $\mathcal{T} \models \alpha \stackrel{t}{\to} \beta$.*

Proof. The “only if” part is easy; just define the satisfaction of sequents by identifying the set on the left side with its conjunction.

For the “if” part, assume that $\mathcal{T}$ does not prove the implication $\gamma \stackrel{s}{\to} \delta$.

Let A be the quotient of $\mathcal{F}_L$ w.r.t. the equivalence relation $\Leftrightarrow$, where $\alpha \Leftrightarrow \beta$ if $\alpha \stackrel{1}{\to} \beta$ and $\beta \stackrel{1}{\to} \alpha$ are provable from $\mathcal{T}$. Then A is naturally endowed with the structure of a De-Morgan lattice.

For $\alpha, \beta \in \mathcal{F}_L$, let $d([\alpha],[\beta]) = 1 - t$, where $t \in [0,1]$ is maximal such that $\mathcal{T} \vdash \alpha \stackrel{t}{\to} \beta$. Endowed with d, A is a metric De-Morgan lattice. By construction, the elements of $\mathcal{T}$ are satisfied by the natural embedding of $\mathcal{F}_L$ into A, but not $\gamma \stackrel{s}{\to} \delta$. □

Needless to comment, our formalism comes closer to measure theory than to fuzzy set theory.

From the interpretational point of view, the semantics based on metric De-Morgan lattices is to be clarified though. Note that only the special case of a Boolean algebra with a submeasure offers an intuitively well comprehensible picture.

4 Conclusion

T-norm-based propositional fuzzy logics are frequently discussed as a suitable tool to model reasoning under vagueness. We have stressed that this is the case as long as the propositions which are formalised share the same reference; namely, they must be modellable by a system of fuzzy sets over a common universe.

If propositions are arbitrary, we run into difficulties when trying to apply techniques of fuzzy logics. This is, for example, the case for the medical expert system Cadiag-2, whose knowledge base contains information on logical and causal relationships between entities which are processed regardless of their meaning. We have risen the question how to define, in this general setting, a minimal frame for what we could call formalised argumentations. As a proposal, we have designed two minimal, but for our needs fully sufficient, systems; we have done so purely syntactically, allowing only rules with a clear interpretation.

The first version concerns reasoning without explicit reference to truth degrees; what comes out is the De-Morgan logic, which allows for interpretations without any connection to fuzziness. The second version incorporates truth values, but the calculus which comes out, still is by no means related to t-norm-based fuzzy logic. The semantics which can be defined ex post are De-Morgan lattices endowed with a non-symmetric distance function.

Our calculi are qualified in that they allow to reproduce the inference mechanism of Cadiag-2. The further elaboration on details of the calculus is work to be done, as well as the analysis of the newly introduced notion of a metric De-Morgan lattice.

References

[1] K.-P. Adlassnig, G. Kolarz, W. Scheithauer, H. Effenberger, G. Grabner, Cadiag: Approaches to computer-assisted medical diagnosis, *Comput. Biol. Med.*, 15:315–335, 1985.

[2] K.-P. Adlassnig, G. Kolarz, W. Scheithauer, G. Grabner, Approach to a hospital-based application of a medical expert system, *Med. Inform.*, 11:205–223, 1986.

[3] A. R. Anderson, N. D. Belnap, J. M. Dunn, “Entailment. The logic of relevance and necessity”, Vol. I and II, Princeton University Press, Princeton 1975 and 1992.

[4] L. Běhounek, On the difference between traditional and deductive fuzzy logic, *Fuzzy Sets Syst.* 159:1153–1164, 2008.

[5] M. Baaz, A. Ciabattoni, C. G. Fermüller, Cut-Elimination in a Sequents-of-Relations Calculus for Gödel Logic, in: Proceedings of the 31st International

ISBN: 978-989-95079-6-8

Symposium on Multiple-Valued Logic (Warsaw 2001), IEEE Computer Society; 181–186, 2001.

[6] B. De Baets, R. Mesiar, Pseudo-metrics and $\mathcal{T}$-equivalences, *J. Fuzzy Math.*, 5:471–481, 1997.

[7] A. Ciabattoni, T. Vetterlein, On the (fuzzy) logical content of Cadiag-2, submitted.

[8] F. Esteva, Ll. Godo, P. Hájek, M. Navara, Residuated fuzzy logics with an involutive negation, *Arch. Math. Logic*, 39:103–124, 2000.

[9] C. G. Fermüller, Theories of vagueness versus fuzzy logic: Can logicians learn from philosophers? *Neural Network World*, 13:455–466, 2003.

[10] C. G. Fermüller, Revisiting Giles's game, in: O. Majer, A.-V. Pietarinen, T. Tulenheimo (eds.), *Games: Unifying Logic, Language, and Philosophy*, Springer-Verlag, Berlin 2009; 209–228.

[11] J. M. Font, Belnap's four-valued logic and De Morgan lattices, *Log. J. IGPL*, 5:413–440, 1997.

[12] D. M. Gabbay, R. H. Johnson, H. J. Ohlbach, J. Woods, *Handbook of the logic of argument and inference. The turn towards the practical*, North-Holland, Amsterdam 2002.

[13] S. Gottwald, *A Treatise on Many-Valued Logics*, Research Studies Press, Baldock 2001.

[14] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht 1998.

[15] H. M. Hersh, A. Caramazza, A fuzzy set approach to modifiers and vagueness in natural language, *Journal of Experimental Psychology: General*, 105:254–276, 1976.

[16] J. A. Kalman, Lattices with involution, *Trans. Am. Math. Soc.*, 87:485–491, 1058.

[17] A. Matoušková, V. Novák, J. Kulka, Psycholinguistic Research of the Meaning of Linguistic Hedges, draft.

[18] V. Novák, I. Perfilieva, J. Močkoř, *Mathematical principles of fuzzy logic*, Kluwer, Dordrecht 1999.

[19] M. Öztürk, A. Tsoukiàs, Modelling uncertain positive and negative reasons in decision aiding, *Decision Support Systems* 43:1512–1526, 2007.

[20] T. Vetterlein, T-norms induced by metrics on Boolean algebras, *Soft Comput.*, 10:995–1000, 2006.

[21] T. Vetterlein, Fuzzy logic as a logic of the expressive strength of information, *Soft Comput.*, 12:479–485, 2008.

[22] T. Vetterlein, A way to interpret Łukasiewicz Logic and Basic Logic, *Studia Logica*, 90:407–423, 2008.

ISBN: 978-989-95079-6-8